The Electronic Journal of Mathematics and Technology, Volume 5, Number 2, ISSN 1933-2823

The Effective Use of LaTeX Drawing in Linear Algebra —Utilization of Graphics Drawn with KETpic—

Masataka Kaneko
nkaneko@inc.kisarazu.ac.jp
Faculty of Fundamental Study
Kisarazu National College of Technology
292-0041
Japan

Setsuo Takato
takato@phar.toho-u.ac.jp
Faculty of Pharmaceutical Science
Toho university
274-8510
Japan

Abstract

The use of geometry (or graphics) to teach and learn linear algebra is an interesting theme. Several published works have studied it, and have illustrated that the use of graphics can be helpful. However, the results of our questionnaire survey (in Japan) and our survey on linear algebra textbooks (in Japanese and English) show that the use of graphics tends to be held off in case of linear algebra compared to the case of calculus or analytic geometry. This tendency is most remarkable in general linear algebra with which the concepts generalizable to the vector spaces other than R^n *(such as the space of functions, matricies, progressions, and so on) are concerned. Persisting in the fact that linear algebra is a general theory to unify various mathematical objects seems to reduce the teachers' incentive for using graphics. However, as shown in this paper, many topics which motivate students to learn abstract concepts of general linear algebra can be found in the geometric (Euclidean) context. Hence, the use of geometric models (or 3D-graphics) should have strongly positive effects on learning them. The aim of this paper is to illustrate that using LaTeX graphics drawn with KETpic should serve a great help for students in their learning linear algebra. KETpic is a macro package of computer algebra systems (CAS) to insert high-quality graphics into LaTeX documents. The 3D-graphics drawn with KETpic are equipped with high accuracy (due to the use of CAS) and rich perspectives. The educational effect of using them will be shown through the result of our students' interview.*

1 Introduction

Based on the historical analysis of linear algebra, J-L. Dorier suggested that "Linear algebra is a general theory designed to unify several branches of mathematics" [1]. In fact, due to the structural isomorphism theorem, any linear transformation on vector space can be identified to a matrix multiplication on Euclidean space R^n (with respect to a specific choice of basis). For this reason, LACSG (Linear Algebra Curriculum Study Group) recommended that both matrix-oriented theory and general theory should be included in the course of linear algebra. Also the process from concrete and practical examples to the development of abstract theory that makes linear algebra so powerful was recommended [2]. Many concrete examples in linear algebra are geometric ones. Therefore it is natural that several published studies have investigated how geometry can be used to introduce the general theory. For example, the work of G. Gueudet [3] comprehensively investigated the effect of using geometry (or geometric models) in teaching and learning linear algebra. Her research ground is Fischbein's theory of intuitions [4] and her research method widely spread to questionnaire survey to 31 mathematicians, research of the textbooks containing that written by T. Banchoff and J. Wermer [5], and interview to 8 students. One of the conclusions is that "Though linear algebra can not be taught nor learned as a mere generalization of geometry, geometric model can be helpful. Especially R^2-R^3 model serves a good paradigmatic model for R^n. However, using figural models for general theory requires additional research". Here the term "general theory" (or "general linear algebra") means the set of concepts which are generalizable to the vector spaces other than R^n (such as linear independence, subspace, basis, dimension, linear maps, and so on).

The aim of this paper is to investigate how geometry (or geometric intuition) can be used to introduce (or motivate to learn) general linear algebra. Since figural models or students' intuition are only applicable to R^2-R^3 model, the possibility or effectiveness of using actual entities in general theory seems to be small. Our research questions are the following:

A: Are there any new possibilities of using graphics to introduce (or motivate to learn) general linear algebra?

B: In which conditions does the use of graphics become effective in students' learning general linear algebra?

Here we take the following strategy. We show students motivating applications of general theory in R^2-R^3 model, instead of geometric (or intuitive) explanation of generalizable concepts. The presentation of motivating examples often requires the use of high-quality graphics. We choose graphics drawn with KETpic as our main tool. KETpic is a macro package of computer algebra system (CAS) to insert fine graphics into LaTeX documents [6, 7, 8]. Now KETpic versions for Maple, Mathematica, Matlab, Maxima, Scilab, and R have been developed. They are freely downloadable from the URL: http://ketpic.com. The graphics generated by KETpic have various features such as

1. Owing to the use of CAS, accuracy in shape and length is guaranteed.
2. Rich mathematical expressions (with the same quality as in LaTeX documents) and various accessories (such as hatchings, shadings, and arrow lines) can be easily inserted into figures.
3. Figures are drawn by using only monochrome lines (i.e. without using colors or shadings) so that the quality of figures is maintained when they are copied [9].
4. 3D-graphics can be drawn with precise shape and rich perspective.

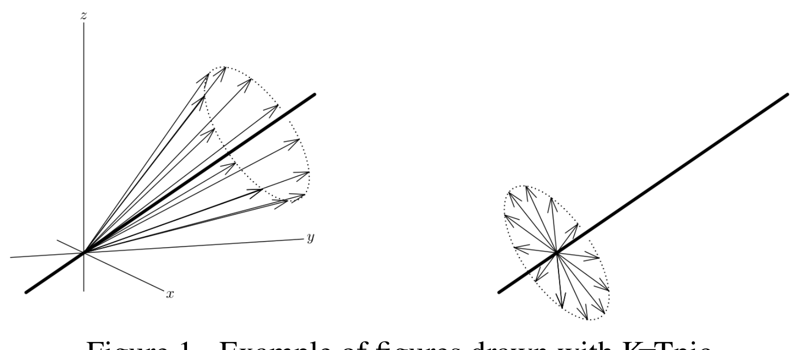


Figure 1. Example of figures drawn with KETpic

For example, the above figures are drawn with KETpic version for Mathematica. They are used to explain the nonexistence of real eigenvectors corresponding to imaginary eigenvalues. We remark that the functionality of hidden line elimination endows these figures with rich perspectives. Moreover, using for-loop in the KETpic programming enables us to easily generate these high-quality graphical images.

Our main proposal is that the ability to produce high-quality graphics improves the teachers' capability to search many illuminative examples which motivate students to learn abstract concepts in general theory. Since graphics of KETpic are coordinated with high-quality mathematical typesetting of LATEX and are suitable for mass printed materials, they should serve powerful tool for teaching general linear algebra.

The results of our questionnaire survey to mathematics teachers in Japan (§2) and research on general linear algebra textbooks (§3) show that the use of graphics in teaching materials and textbooks of linear algebra is tend to be held off. It is also shown that this tendency is most remarkable in case of general linear algebra. In §4, we analyze the reason of this and related problems based on the theoretical framework which is suggested by G. Harel [10]. Then we will present some examples of our teaching materials or some parts of our textbook in general theory. They contain graphics drawn with KETpic (§5). Last we will demonstrate the effect of using graphics drawn with KETpic through the result of our interview to 8 students at a college of technology in Japan (§6). Some difficulties are also commented (§7).

2 Questionnaire survey

In this section, we show some part of the contents and results of our questionnaire survey. This survey had been carried out (in 2008) to investigate the following points:

1. The methods how teachers produce and use graphics
2. The needs of teachers for using graphical class materials

The questionnaires were posted to teachers at universities and college of technologies in Japan. 667 teachers (of mathematics, computer science, physics, technology, etc.) at 23 universities and 56 college of technologies answered. Here we analyze only the answers of 378 mathematics teachers. Among them, the answers of 232 mathematics teachers at college of technology have already been analyzed in [6].

Q.1 Frequency of using other device than blackboard to display graphics

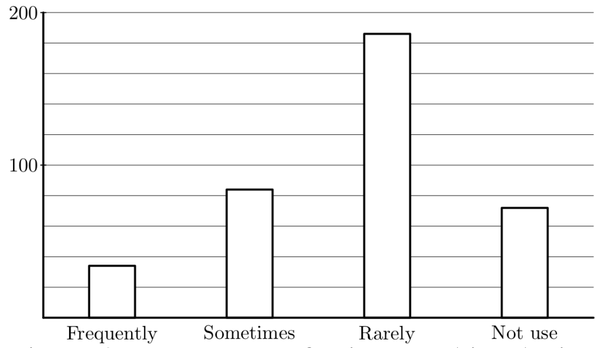


Figure 2. Frequency of using graphics device

Q.2–5 are questions to those who answered "frequently" or "sometimes" in **Q.1**.

Q.2 Subjects with frequent display of graphics (multiple answers allowed)

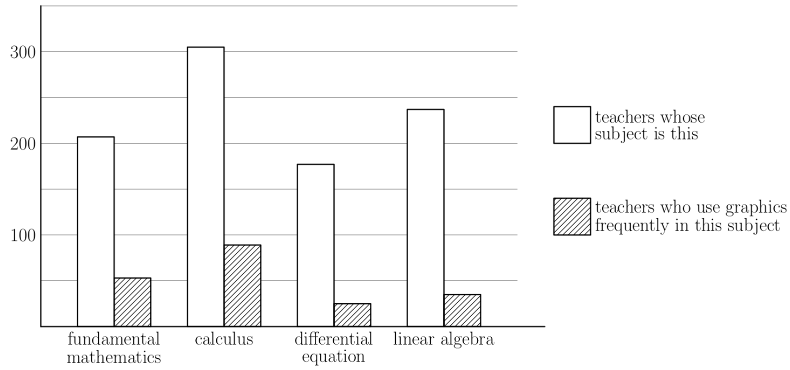


Figure 3. Use of graphics in various subjects

In case of linear algebra and differential equations, the use of graphics turns out to be held off.

Q.3 Method to generate graphics (multiple answers allowed)

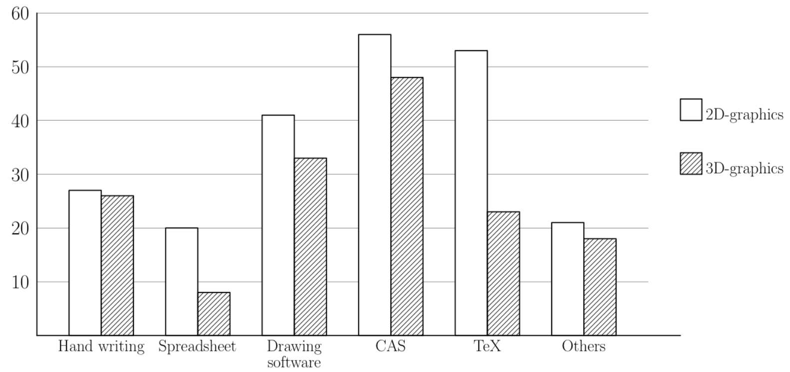


Figure 4. Method to generate graphics

Generating 3D-graphics is shown to be a difficult task for teachers especially in case of spreadsheet and TEX-drawing.

Q.4 Method to display graphics (multiple answers allowed)

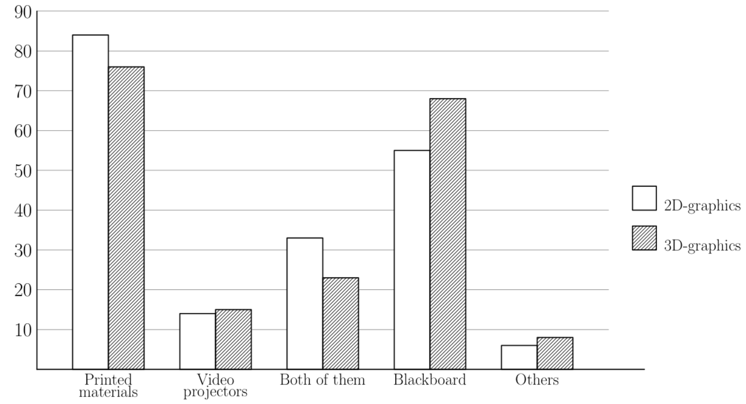


Figure 5. Method to display graphics

Printed matter is the most frequent choice. Compared to the case of 2D-graphics, writing on blackboard is more preferred in case of 3D-graphics.

Q.5 Which topics the display of graphics is effective for? (at most three topics)

The answers are classified to the following two categories:

1. Topics needing precise figures
   (a) Graph of one or two variable functions (77)
   (b) Taylor series expansions (26)
   (c) Solution curves for differential equation (9)
   (d) Quadratic curves and surfaces (4)
2. Topics needing conceptual figures which are difficult to be drawn on blackboard
   (a) Differential and Integral (27)
   (b) Partial differential and Total differential (24)
   (c) Double integral and Repeated integral (23)
   (d) Vectors (13)

In particular, only three teachers answered the topics related to general linear algebra. They answered the visualization of linear transformations and one of them answered the explanation of eigenvalues and eigenvectors.

Q.6–7 are questions to those who answered “rarely” or “not use” in **Q.1**.

Q.6 Frequency of drawing figures on blackboard

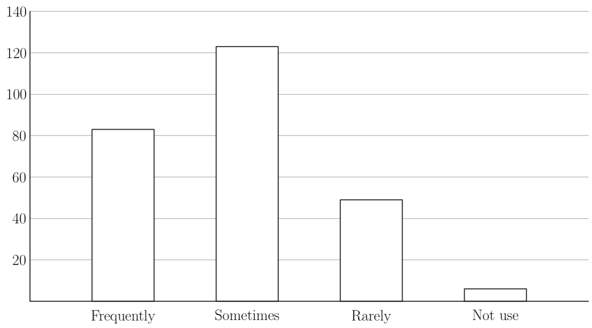


Figure 6. Use of blackboards

Blackboard is shown to be the most frequent choice of teachers among all devices to show graphics.

Q.7 (To those who answered “frequently” or “sometimes” in **Q.6**)
What prevents them from using other device than blackboard

(multiple answers allowed)

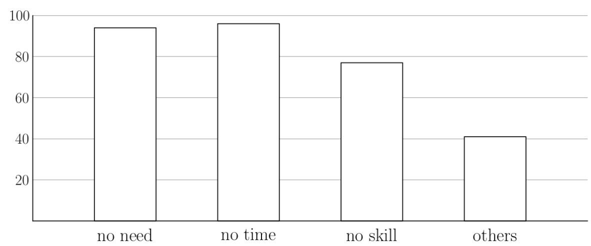


Figure 7. The reasons of not using graphics device

There seems to be great potential needs for devices which enable teachers to genarate graphical images quickly and easily in their teaching materials.

3 Textbook research

Some of the teachers who answered "Rarely" or "Not use" to **Q.6** in the questionnaire claim that they feel no need for using garphics in their teaching materials or their writing on blackboard since using figures in textbooks is sufficient for their teaching. Therefore we also executed the research on the use of graphics in linear algebra textbooks.

First we compared the frequency of using graphics in linear algebra textbooks with that in calculus textbooks. This comparison was made between four pairs of textbooks. Each pair belongs to the same series of Japanese popular mathematics textbooks respectively. Figure 8 shows the number of figures used in each textbook.

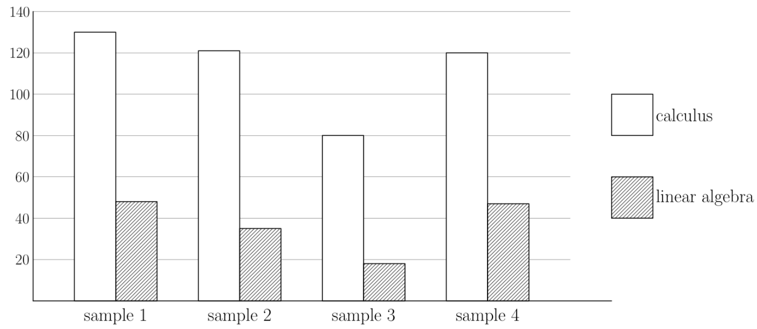


Figure 8. Comparison between calculus and linear algebra

As seen in Figure 8, the frequency in linear algebra textbooks is much lower than that in calculus textbooks. This tendency coincides with the result of **Q.6** in our questionnaire survey (see Figure 3).

To clarify the possibilities and limitations of using R^2-R^3 model, G. Gueudet researched various textbooks. She picked up the textbook written by T. Banchoff and J. Wermer [5] and pointed out the following:

1. Thanks to the use of coordinates, the generalization from the R^2-R^3 model to R^n (especially to R^4) has been accomplished explicitly. Many drawings are used to explain various notions in R^2-R^3 model and R^n.
2. The generalization from R^n to general theory has been accomplished by using R^n as a new paradigmatic model. Almost no drawings are used in the part of general theory since R^2-R^3 model is not used as a paradigmatic model of general theory.

In fact, only two figures in Figure 9 are used in the part of general theory. The left figure is used to explain basis of R^n, and the right figure is used to explain orthogonal decomposition.

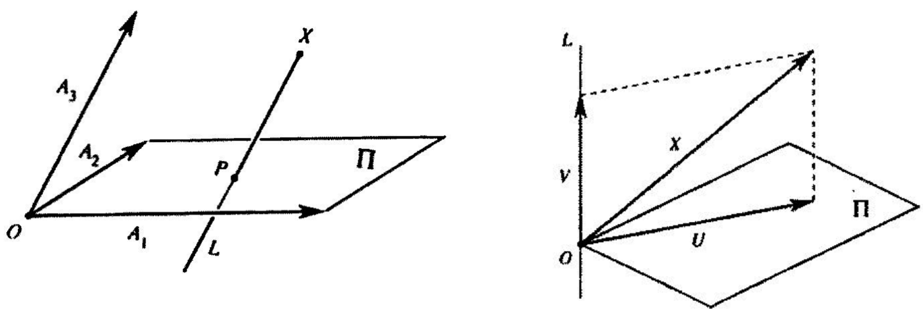


Figure 9. Figures used in general theory context (Banchoff-Wermer)

To investigate the use of graphics in general linear algebra textbooks more widely, we surveyed some other English linear algebra textbooks which contain general theory. Then it has turned out that almost all of them use few graphics. One exception is the textbook written by S. Lang [11]. As shown in Table 1, the use of graphics in Lang's textbook has a similar feature to that in Banchoff-Wermer's one in that a small number of figures are used in general theory context. However there are some differences in the following two points:

1. R^2-R^3 figures are used in general theory context.
2. The use of R^3 figures is fairly limited.

Table 1. The use of figures in Lang's textbook

Contents	Number of figures	Comments
vector space and matrix	28	R^2-R^3 model
linear maps and matrix	12	general theory with R^2 figure
inner products	1	R^2 figure
determinants	10	R^2-R^3 model
eigenvalues and eigenvectors	2	R^2 figure
convexity	5	R^n context with R^2 figure

As another example of textbook which uses figures aggressively in the general theory context, we pick up the textbook written by G. Strang [12]. Even in this case, only three figures are used in general theory context. All of them are shown in Figure 10.

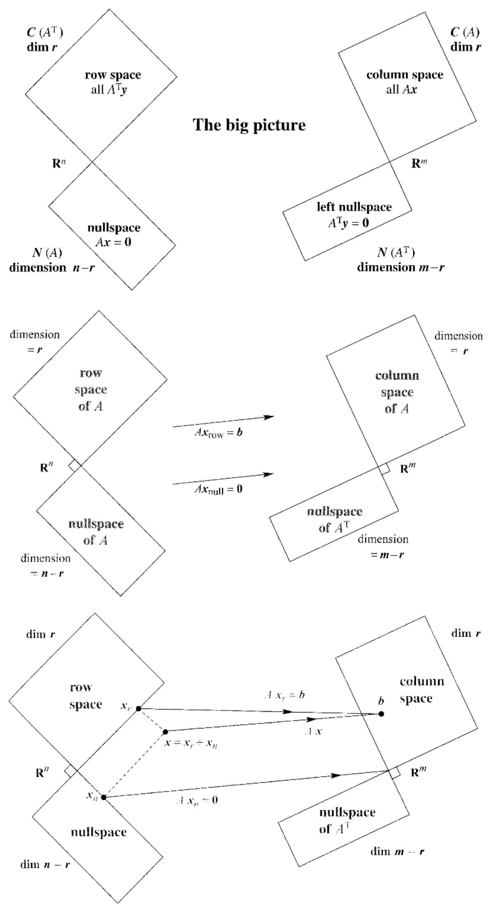


Figure 10. Figures used in general theory context (Strang)

We remark that all figures in Figure 10 are used to explain the concepts of general theory (structure theorem of linear maps and dimensions of kernel and cokernel) in terms of R^n model. So they are directly connected to matrix oriented linear algebra. Also we remark that they are abstract figures since R^n model can not be realized in R^2-R^3 model. Moreover elements of a vector space are represented as points. This is quite different from the case of Banchoff-Wermer's textbook where elements are represented as arrows. Therefore the above figures are out of the "familiar geometry" and some process of abstraction is needed for students to fully understand the meaning of Figure 10.

Though Figure 9 and 10 are precise figures in a sense, the meaning of "preciseness" seems to be different from the cases of R^2-R^3 model (analytic geometry), calculus, and differential equations. This is because these figures are not associated with specific examples. So that, it will be possible for teachers to draw these figures on blackboard with almost the same quality.

4 Theoretical framework and analysis

Though previous researches have indicated that using figures (or geometric models) is great help for students in their learning linear algebra, the results in §2 and §3 show that the use of figures (or geometric models) in teaching materials and textbooks tends to be insufficient. In my experience of teaching linear algebra, this insufficiency poses various difficulties in students' learning linear algebra. In this section, we analyze the following two points:

1. What is the reason of insufficient use of figures (or geometric models)?
2. What sort of difficulties arises from this insufficiency?

The analysis is based on the theoretical framework suggested by G. Harel [10]. He presented three necessary conditions for teaching and learning mathematical concepts as didactical principles. First we will revise them briefly.

A. Concreteness

> For students to abstract a mathematical structure from a given model, the elements of that model must be conceptual entities in the students' eyes.

Following this principle, using R^2-R^3 model to present basic concepts of linear algebra at the first stage is recommended. Also, in the next stage, using vector spaces of dimension less than or equal to 3 with general elements is encouraged, so that students can absorb the idea that derived results of linear algebra depend solely on the axioms of vector space, not upon the definitions of specific elements.

B. Necessity

> For students to learn, they must see a need for what they are to be taught. 'Need' means an intellectual need, as opposed to a social or economic need.

Following this principle, teachers are encouraged to provide students with problem-solving activities where the learner applies existing conceptions to solve problems and modifies these conceptions when encountering cognitive conflicts. The idea behind this principle is that instructional environments must include appropriate constraints by which students can reflectively abstract mathematical conceptions and, at the same time, keep the situation at hand realistic. It is pointed out that, through their activities, students must feel what they do results in the solution of problem or conflict.

C. Generalizability

> When instruction is concerned with a concrete model, the instructional activities within this model should allow and encourage the generalizibility of concepts.

This principle complements the above two principles. Especially it is emphasized that the application of generalizability principle must be in accordance with the necessity principle. Otherwise, students feel no intellectual need for creating new concept.

To satisfy concreteness principle, teachers must endow the elements of their classroom with reality. One typical method for that should be using graphics. Regarding the above principles, the following conclusions will be induced from our considerations in §2 and §3.

1. To satisfy generalizability principle, some process of formalization should be inevitable. Axiomatic approach will be the most typical methodology. Since general linear algebra is a theory designed to unify several branches of mathematics, the use of axiomatic approach is required more frequently compared to the case of calculus or analytic geometry. Though graphics can be utilized in case of R^2-R^3 model, many basic concepts in general linear algebra become self evident in that case.

2. To satisfy necessity principle, teachers must give students motivation for them to learn a new concept. For that purpose, illuminative application of that concept is desirable. To present such application, high-quality graphics are often needed. Though 3D-graphics are more preferred than 2D-graphics to inspire students' motivation, drawing high-quality 3D-graphics is not so easy task for teachers.

Thus teachers seem to be prevented from using appropriate graphics in order to present motivating applications of generalizable concepts. The lack of graphics should result in the lack of reality to the students' eyes. In this way, serious difficulties arise in students' learning general linear algebra. They will not feel necessity to learn generalizable concepts without reality. Moreover, the result of **Q.7** in §2 indicates that teachers' insufficient capability to draw graphics may restrict their possibility to choose illuminative topics.

5 Utilization of graphics drawn with KETpic

In this section, we will show some graphics drawn with KETpic contained in our teaching materials or textbook in general linear algebra.

The first example is the figure used to explain the meaning of eigenvalues and eigenvectors. In many textbooks, 2D-graphics like Figure 11 are often used.

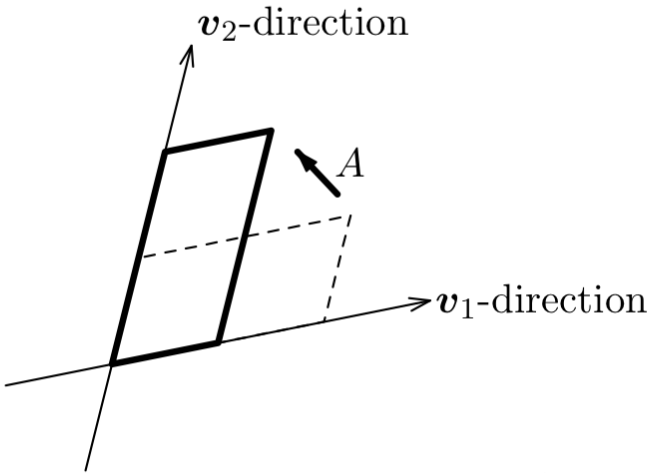


Figure 11. Eigenvalues and eigenvectors (2D version)

However, in 2D-graphics case, students may feel that they can obtain fairly good understanding about the structure of linear transformation on $\mathbf{R}^2$ by seeing graphics like Figure 12. So some students may not be convinced of the value to consider eigenvalues and eigenvectors.

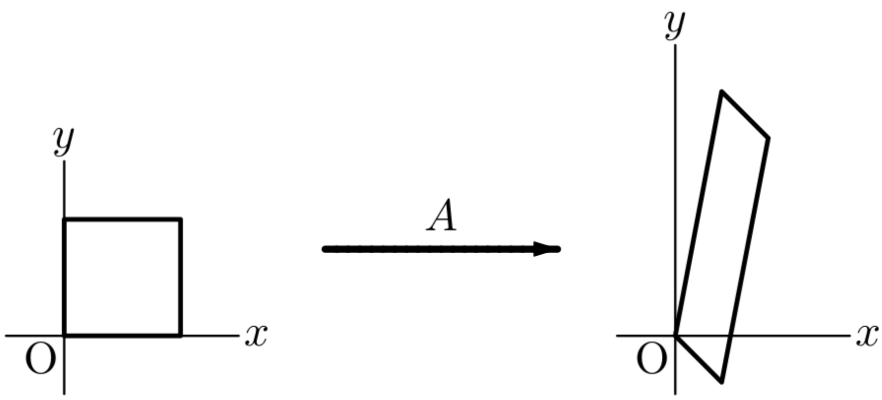


Figure 12. Canonical basis and their images (2D version)

Therefore we have been using the 3D-graphics counterparts of these figures like Figure 13 in our general linear algebra textbook.

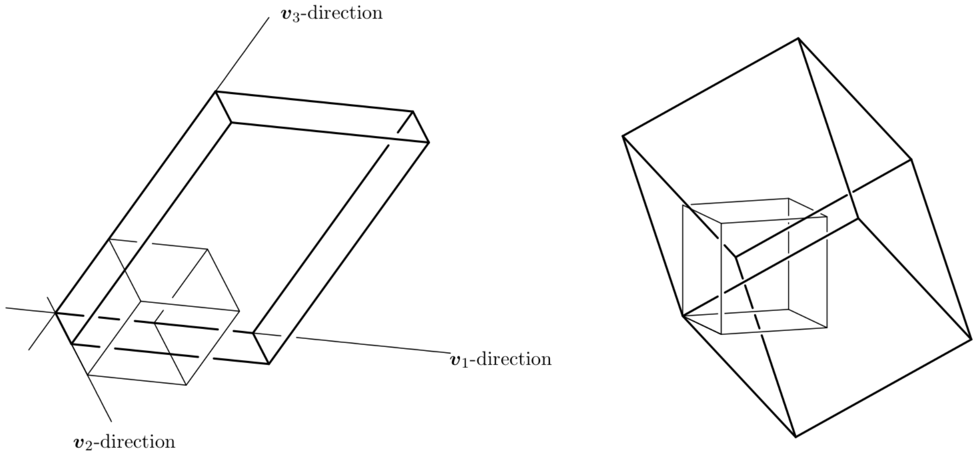


Figure 13. 3D-graphics counterparts of Figure 11 and 12

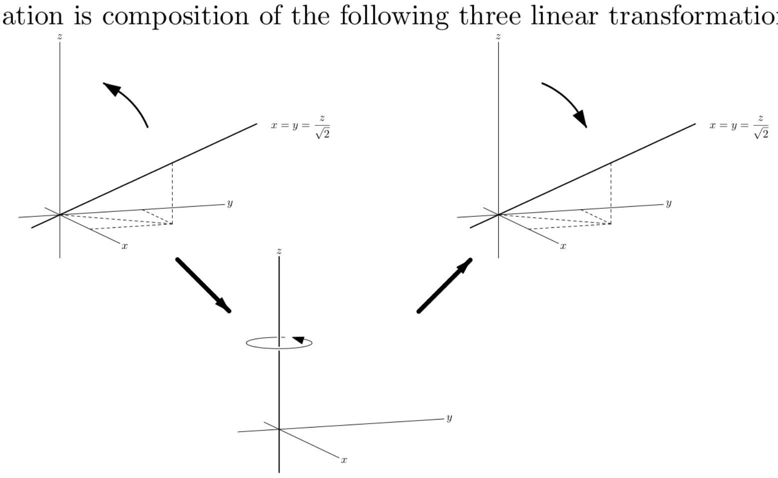
The second example is an exercise in our textbook used to apply the composition of linear transformations. It is also related to conjugation.

Problem

Let l be the space line $x = y = \dfrac{z}{\sqrt{2}}$. Compute the matrix giving the rotation around l by the angle θ.

Hint

The rotation is composition of the following three linear transformations.



We remark that precise image of spatial rotation can be generated by using only single command "`rotate3data`" equipped to KETpic. The programmability of KETpic like this enables teachers to produce their teaching materials with precise figures quickly and easily.

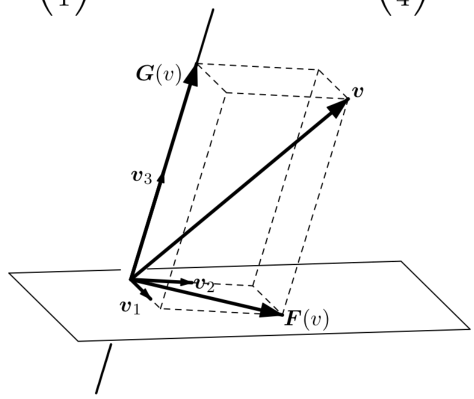
The third example is also an exercise in our textbook concerning parallel projection. The themes are the change of basis and matrix representation of a linear transformation.

Problem

Let l be the line $\dfrac{x-5}{2} = \dfrac{y-6}{3} = \dfrac{z-7}{4}$, and π be the plane $2x + y - 2z = 1$. Compute the matrix giving the parallel projection $\boldsymbol{F}$ onto l along π, and the projection $\boldsymbol{G}$ onto π along l.

Hint

$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ generate π, and $\boldsymbol{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ generates l.



Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are basis of $\mathbf{R}^3$, it is sufficient to calculate the images of them.

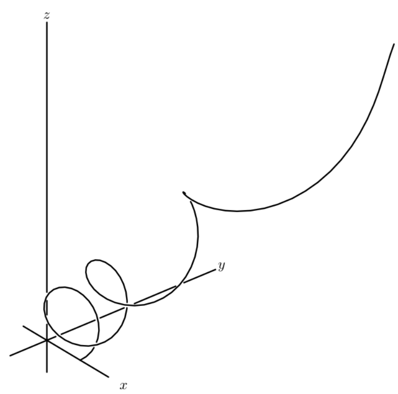
In this case, preciseness of the figure is essential. When I draw this figure on blackboard, many students confused this problem with that of Gram-Schmidt method. Using printed materials containing this figure seems to keep students from such confusion.

The last example is a teaching material prepared for tutorials. The theme is to investigate the nature of the solution curve for a differential system. We aimed to let students understand the meaning of the change of basis in case of normal form (i.e. other than diagonalization).

Problem

The solution curve of the following simultaneous differential equation is shown below.

$$\begin{cases} \frac{dx}{dt} = x - y + z & x(0) = \sqrt{2} \\ \frac{dy}{dt} = x \qquad - z & y(0) = 0 \\ \frac{dz}{dt} = x + y + z & z(0) = 0 \end{cases}$$



Show that this solution curve is located on a cylinder.

Solution

Since the equation can be expressed in the following matrix form:

$$\frac{d}{dt}\overrightarrow{p} = A\overrightarrow{p} \qquad \overrightarrow{p}(0) = \overrightarrow{p_0}$$

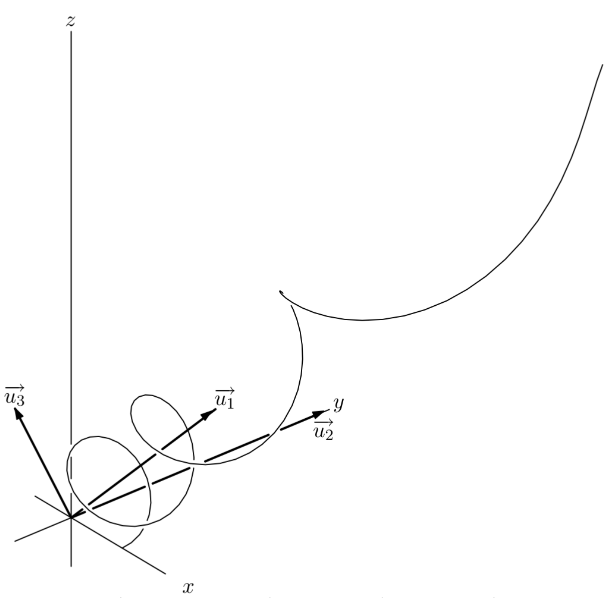
$$\overrightarrow{p}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \overrightarrow{p_0} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

the solution is given as $\overrightarrow{p}(t) = \exp(At)\overrightarrow{p_0}$. If we put $T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$,

then $T^{-1}AT = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$ holds. Thus we put

$$\overrightarrow{u_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \overrightarrow{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \overrightarrow{u_3} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

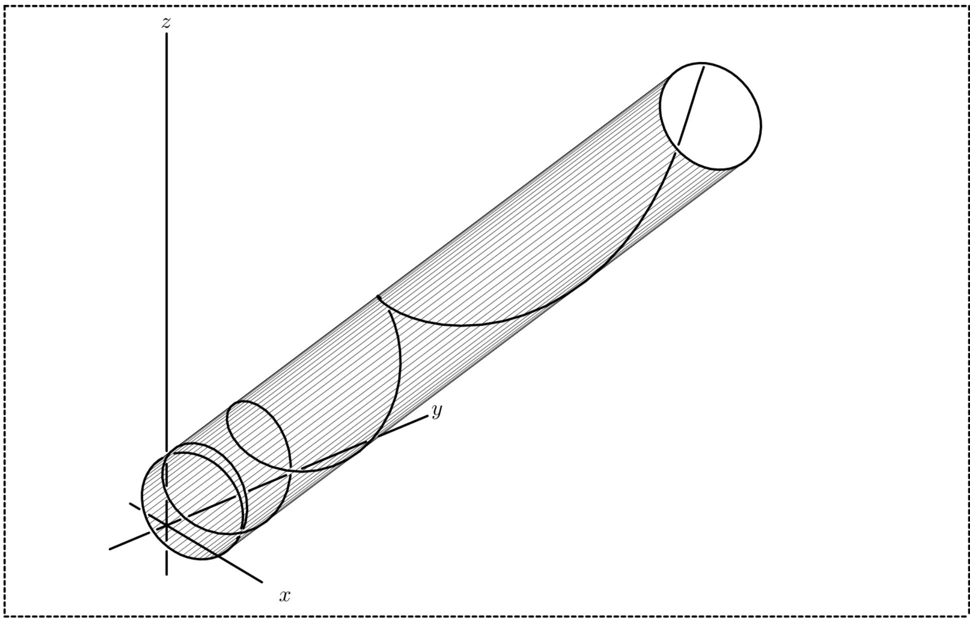
as shown in the following figure:



If we express $\overrightarrow{p}(t) = X(t)\overrightarrow{u_1} + Y(t)\overrightarrow{u_2} + Z(t)\overrightarrow{u_3}$, the following equality holds:

$$\begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & \cos\sqrt{2}t & -\sin\sqrt{2}t \\ 0 & \sin\sqrt{2}t & \cos\sqrt{2}t \end{pmatrix} \begin{pmatrix} X(0) \\ Y(0) \\ Z(0) \end{pmatrix}$$

Therefore, the solution curve is located the cylinder drawn below.



The examples in this section will illustrate that using LaTeX graphics drawn with KETpic enables teachers to produce motivating applications of generalizable concepts, owing to the preciseness and programmability of KETpic drawing and rich perspectives in the 3D-graphics of KETpic.

6 Students' interview

Last, we will show the contents and results of our interview to students. The aim of this interview is to examine the effect of using LaTeX graphics drawn with KETpic in teaching and learning general linear algebra.

The subjects of the interview are 8 students at Kisarazu National College of Technology in Japan. They belong to the last grade (i.e. all of them are 20 years old). They had learned basic theory of vectors in R^3 (containing outer products of vectors) and matrix-oriented linear algebra (containing the calculation of eigenvalues and eigenvectors) in the previous years. Though they had acquired some "knowledge" or "method for computation" about basic concepts in general theory through a series of lectures (25 hours or so) in this year, it was uncertain whether they had appreciated the "meaning" (or "value") of considering such concepts or not. All but the last examples in §5 are contained in the textbook which was used in the lectures. The interview was executed in the two hours of extra lecture.

Following the suggestion of G. Harel, we executed the interview in a problematic situation. The object of the extra lecture is to let students appreciate the meaning of basis change and its influence to matrix representation. The topic chosen is the structure of a linear transformation which is not diagonalizable (i.e. the case when Jordan normal form appears). The scheme for examining the effect is to compare the students' reasoning process before the presentation of graphics with that after the presentation. More precisely, the flow of the interview is as shown below.

First, students are asked to answer the following preliminary problem. To help students' reasoning, Figure 14 was distributed in the form of printed materials.

Preliminary Problem

Let A be the matrix $\begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}$.

(1) Let $A(D)$ be the image of a domain D under the linear transformation given by A. Calculate the ratio of volumes of D and $A(D)$.
(2) Calculate the eigenvalues and eigenvectors of A.
(3) Based on the above results, consider the relationship between $\det A$ and the eigenvalues of A.

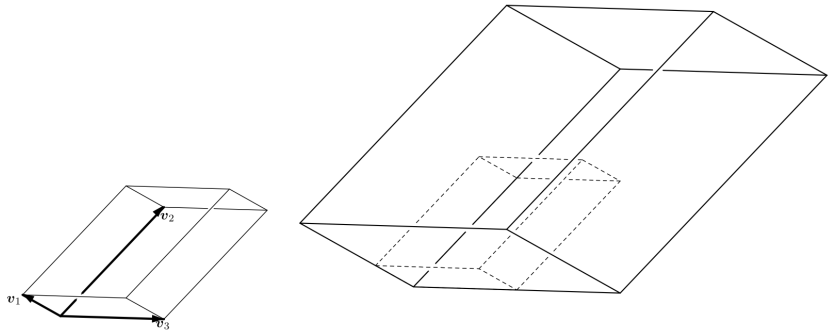


Figure 14. Hint to the Preliminary Problem

Since Figure 14 is precise in lengths and shapes, students can obtain intuitive image for the structure of this transformation by comparing its eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ with their images. Thus all students gained correct solutions in 20 minutes. Teacher explains the solution by using Figure 15 presented both on printed materials and screen of video projector.

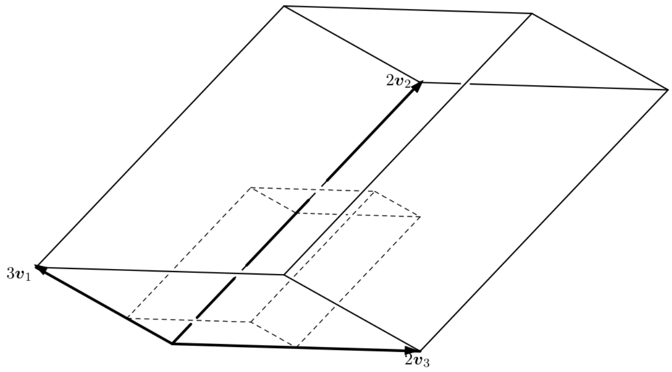


Figure 15. Explanation of solution

Next, students are asked to answer the following main problem. This problem was arranged so that the linear transformation has the same eigenvalues (i.e. $\lambda_1 = 3,\ \lambda_2 = \lambda_3 = 2$) and is not diagonalizable. Moreover, the eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ and eigenvector in the wider sense $\boldsymbol{v}_3$ can be taken to be the same as eigenvectors in the preliminary problem. Therefore, students can utilize Figure 15 to solve the main problem.

Main Problem

Let B be the matrix $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & -1 & 3 \end{pmatrix}$.

(1) Let $B(D)$ be the image of a domain D under the linear transformation given by B. Calculate the ratio of volumes of D and $B(D)$.
(2) Calculate the eigenvalues and eigenvectors of B. Show that B is not diagonalizable.
(3) Determine the eigenvalue λ of B and corresponding eigenvector $\boldsymbol{v}$ such that the vector $\boldsymbol{v}'$ satisfying the following equation exists:

$$(B - \lambda I)\boldsymbol{v}' = \boldsymbol{v}$$

(4) Show that the two eigenvectors of B in question (2) and the vector $\boldsymbol{v}'$ in question (3) serve a basis of R^3.
(5) Calculate the representation matrix of the linear transformation given by B with respect to the above basis of R^3.
(6) Based on the above results, consider the relationship between $\det B$ and the eigenvalues of B.

The result of (5) is Jordan normal form, and the method of the computation leading to Jordan normal form had already been presented to the students in previous lectures. So almost all the students did not take much time to obtain the correct solution of (1)–(5). The highlight is (6). Though they had been given some knowledge about the influence of basis change to representation matrix, they had never been given systematic explanation or motivating examples of it.

As a result, only two students drew appropriate figures and realize the structure by their own efforts.

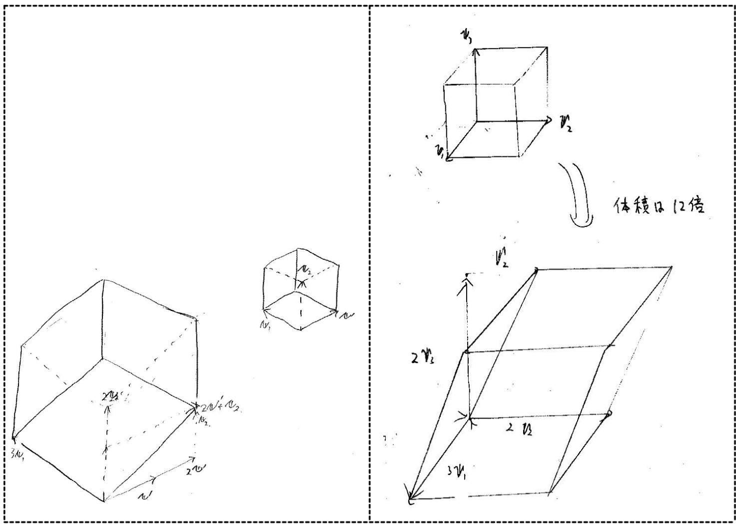


Figure 16. Examples of correct answer

They had already appreciated that the specific choice of basis is not important and expressed the structure in case when the basis is almost orthonormal. Other six students could not realize the structure for 30 minutes. The result of this interview indicates that it can not be expected for usual students to readily understand the meaning of basis change through the curricula focused on the proficiency in matrix computations or the formalization of abstract concepts.

Two of the remaining students drew the following figures.

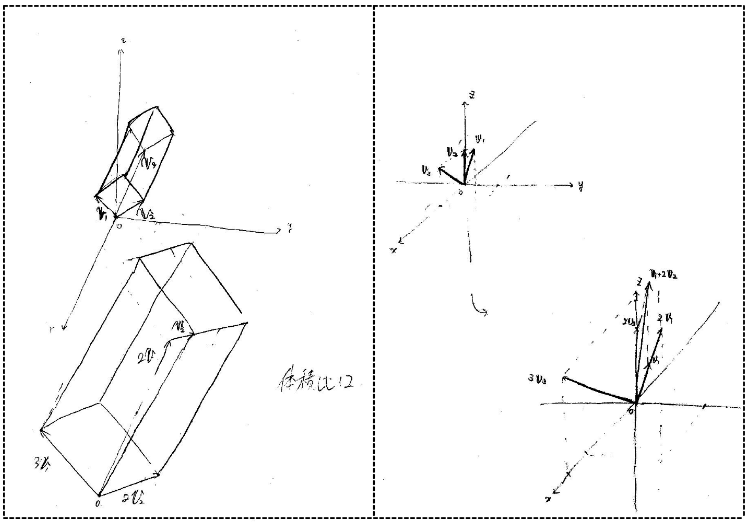


Figure 17. Other examples of students' drawings

They correctly understood matrix representation with respect to a basis. They used the basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ honestly, and could not realize that the distortion in the $\boldsymbol{v}_3$ direction keeps the height of the parallel hexahedron. This result indicates that it is not so easy for usual students to draw appropriate 3D-graphics by hand even in the case of simple shape as above.

The remaining four students could not draw figures or persisted in matrix-oriented calculation as in Figure 18 (originally in Japanese).

Since there is a regular matrix P such that $P^{-1}BP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$,

$|B|$ is equal to the determinant of $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

$$\begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 12$$

Determinant of triangular matrix = Product of diagonal components

Upper triangular components do not affect determinant

(though they affect the shape of the hexahedron).

⇓

The ratio of the volume of the hexahedron is computed to be $\lambda_1\lambda_2\lambda_3$

Figure 18. Reasoning persisted in matrix-oriented calculation

Last, the printed materials containing Figure 19 are distributed to students and students are encouraged to re-try (6). The teacher gave no other hints.

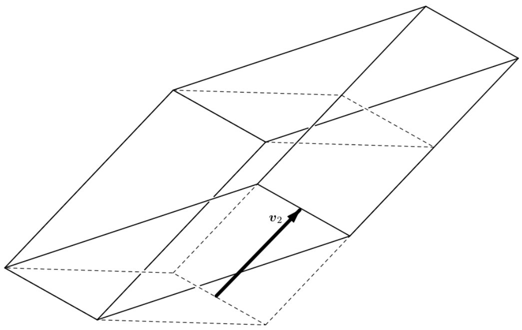


Figure 19. Final hint

Then all the students could appreciate the structure in 10 minutes. Figure 20 shows an example of students' final answers.

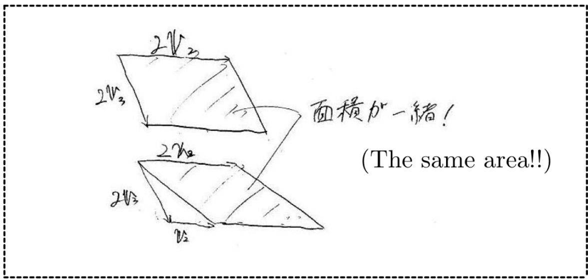


Figure 20. Example of final answer

As is pointed out in [3], many general results can be established by reasoning in a well-chosen 2-dimensional subspace. The sample shown above is consistent with that remark. Transparency in the 3D-graphics of KETpic will become some help to such kind of reasoning.

We also executed a hearing of the students' comments about this lecture after they obtained the correct solution. Then the students responded as follows:

1. By seeing the arrow of $\boldsymbol{v}_2$, it becomes easier for me to realize the relationship to the preliminary case.
2. Adding the arrow of $\boldsymbol{v}_2 + 2\boldsymbol{v}_3$ is desirable.
3. More of the hidden line elimination is desirable.
4. In case when more than one eigenvectors in the wider sense appear, how will the structure become?
5. Without seeing the figure, it should have taken much more time for me to understand the structure.

The fact that the students' reasoning process suddenly changed when Figure 19 was presented illustrates that using precise LaTeX graphics should serve great help for students' reasoning. Also the students' response like (1)-(2) shows that 3D-graphics of KETpic with rich perspectives leads the students' reasoning into the correct direction. It would not be so easy for usual mathematics teachers to draw figures on blackboard with almost the same quality.

Moreover, the students' response like (4) shows that the ability to draw illuminative figures might enhance the teachers' competency to satisfy students' intellectual needs. As an example, we will show Figure 21 which is distributed for further study.

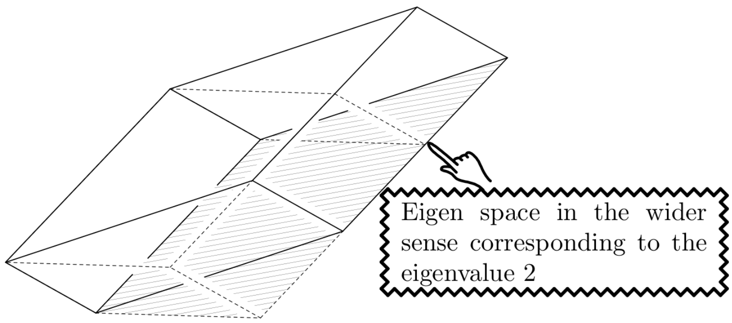


Figure 21. Figure for further study

7 Limitations and future works

Though the graphics drawn with KETpic could improve the possibility of using geometric models in teaching and learning general linear algebra, there are various limitations to that possibility at this time. In this section, we remark the following two points.

7.1 The nature of linear algebra

One of the most serious difficulties which students encounter in their learning linear algebra is that highly sophisticated reasoning process is required to unify various kind of mathematical objects. It is after such reasoning process is completed that students really appreciate the need to learn general linear algebra. The unification is quite different from geometry. This seems to be consistent with the statement made by G. Gueudet [13]: "A geometric model alone seems insufficient to justify the need for a general theory". What we demonstrated in this paper is limited to giving some motivating "examples" in "$\mathbf{R}^2$-$\mathbf{R}^3$ model". Whether our methodology is applicable to justifying the need mentioned above or not requires further research.

7.2 The condition for effective use

The other limitation of our strategy is deeply related to our research question B presented in §1. After the interview in §6, the student who answered as in Figure 18 commented as follows:

> Since I have learned the geometric meaning of eigenvalues and eigenvectors (using Figure 13) before this lesson, I could use Figure 19 to solve the main problem in some way. But it seems to be very difficult for beginning students (of general theory) to understand the whole meaning of this figure.

This comment should indicate that some training is necessary for teachers and students so that the graphics drawn with KETpic can be used effectively in teaching and learning general linear algebra. The necessary condition for such effective use should also be studied.

Acknowledgements

This work is supported by **Grant-in-Aid for Scientific Research(C) 20500818**. The authors are also indebted to Professor Ghislaine Gueudet-Chartier. She read the entire manuscript of this paper and made many valuable comments and suggestions that have improved the quality of this paper.

References

[1] J-L. Dorier, *Epistemological analysis of the genesis of the theory of vector spaces*, in *On the teaching of linear algebra*, Kluwer, Dordrecht, 2000, pp.1–81.

[2] D. Carlson, C.R. Johnson, D.C. Lay, and A.D. Porter, *The linear algebra curriculum study group recommendations for the first course in linear algebra*, The college mathematics journal, 24-1, 1993. pp. 41–46.

[3] G. Gueudet, *Using geometry to teach and learn linear algebra*, in *Research in collegiate mathematics education IV*, American mathematical society, 2006, pp. 171–195.

[4] E. Fischbein, *Intuition in science and mathematics*, Kluwer, Dordrecht, 1987.

[5] T. Banchoff and J. Wermer, *Linear algebra through geometry, 2nd ed.*, Undergraduate texts in math, Springer, 1992.

[6] K. Kitahara, T. Abe, M. Kaneko, S. Yamashita, and S. Takato, *Towards a more effective use of 3D-graphics in mathematics education – Utilization of KETpic to insert figures into LaTeX documents –*, to appear in IJTME. 17-4 (2010).

[7] H. Koshikawa, M. Kaneko, S. Yamashita, K. Kitahara, and S. Takato, *Handier Use of Scilab to Draw Fine LaTeX Figures – Usage of KETpic Version for Scilab –*, Proc. ICCSA 2010, IEEE Press, pp. 39–48.

[8] M. Kaneko, T. Abe, K.Fukazawa, M.Sekiguchi, Y.Tadokoro, S. Yamashita, and S. Takato, *CAS-aided visualization in LaTeX documents for mathematical education*, TMCS. 8-1 (2010), pp. 1–18.

[9] M. Kaneko, T. Abe, H. Izumi, K. Kitahara, M. Sekiguchi, Y. Tadokoro, S. Yamashita, K. Fukazawa, and S. Takato, *A simple method of the TeXsurface drawing suitable for teaching materials with the aid of CAS*, Proc. ICCS 2008, L.N.C.S. 5102, Springer, pp. 35–45.

[10] G. Harel, *Three principles of learning and teaching mathematics*, in *On the teaching of linear algebra*, Kluwer, Dordrecht, 2000, pp. 177-189.

[11] S. Lang, *Introduction to linear algebra, 2nd ed.*, Undergraduate texts in math, Springer, 1986.

[12] G. Strang, *Introduction to linear algebra, 4th ed.*, Wellesley-Cambridge Press, 2009.

[13] G. Gueudet, *Should we teach linear algebra through geometry?*, Linear algebra and its applications 379 (2004), Elsevier, pp. 491–501.